GENERALIZED OPERATORS AND SYMBOLIC COMPUTATION FOR NONLINEAR SYSTEMS

Miroslav Halás

Department of Automation and Control
Faculty of Electrical Engineering and Information Technology
Slovak University of Technology
Ilkovičova 3, 812 19 Bratislava, Slovakia
miroslav.halas@stuba.sk

Abstract: In linear systems Laplace transformation plays an important role. It enables us to design controllers, switch between different types of system representations. For those purposes we use well known transfer functions.
Obviously, the situation is different when systems are nonlinear. The main problem, which makes the analysis and synthesis of such systems difficult, is an invalidity of basic principles employed in linear systems. Above all, it is the invalidity of the associativity which disables us to employ a linear theory and transfer functions. Nevertheless, an algebraic point of view, which presented paper introduces, enables to define a similar symbolic computation. From that point of view differential and derivative operators play a key role. In terms of the introduced symbolic computation the paper tries to depict a solution to a few basic control problems, namely a problem of modeling and a control design for nonlinear systems.

Keywords: nonlinear system, algebraic approach, differential operators.

1. INTRODUCTION

Solutions to the nonlinear problems employing linear methods have origins in the 19th century. The Ljapunow work on linearizing system behavior around a fixed operating point still represents a footstone of the present nonlinear control theory. Also the feedback linearization (Isidori, 1989) or the algebraic approach (Conte, *et al*., 1999) should be mentioned.
The main problem, which makes the analysis and synthesis of nonlinear systems difficult, is an invalidity of some basic principles typically employed in linear systems. Mainly the principle of the superposition is invalid. This fact, in general, disables to use the classical linear-system-based approach, covering the transfer functions, for nonlinear systems. Presented paper tries to avoid the problem by employing differential operators. Some possibilities are depicted in (Halás *et al*., 2003). Obviously, the general solution to the problem will require a complex approach. This work represents an outline of the problem formulation and several hypothesis that should be yet pinpoint by rigorous proofs.
The paper is organized as follows. In §2, the basic ideas of (Halás *et al*., 2003) are depicted. Preliminaries of an algebraic approach are reviewed in §3. The symbolic computation of

nonlinear systems is discussed in §4 and applied to a few basic control problems, namely modeling and control design, in §5. Finally, conclusions are summarized in §6.

2. PRINCIPLES OF THE EXACT VELOCITY LINEARIZATION METHOD

In this section the basic idea of the exact velocity linearization method (Halás *et al.*, 2003) is depicted. Instead of the velocity forms the differential forms are employed.

The question can be asked as to whether it is possible to find an approach allowing to introduce transfer functions of nonlinear systems which would provide both the possibility to use a transfer function algebra and also the possibility to characterize nonlinear dynamics. To answer this question let us consider two simple nonlinear systems

$$\begin{aligned} y_1 &= g_1(u_1) \\ y_2 &= g_2(u_2) \end{aligned} \tag{1}$$

where g_1, g_2 are nonlinear differentiable functions. The principle of the superposition is invalid when systems are nonlinear, as generally known. Obviously, the system $y = g_1(g_2(u))$ does not equal the system $y = g_2(g_1(u))$. However, we can formally avoid this problem by differentiating, since a derivative of a composite function is simply a product of derivatives of its components. Thus on differentiating (1)

$$\begin{aligned} \mathrm{d}y_1 &= K_1 \mathrm{d}u_1 \\ \mathrm{d}y_2 &= K_2 \mathrm{d}u_2 \end{aligned} \tag{2}$$

where $K_1 = \partial g_1 / \partial u_1$ and $K_2 = \partial g_2 / \partial u_2$. Now, formally

$$\mathrm{d}y = K_1 K_2 \mathrm{d}u \tag{3}$$

for both $y = g_2(g_1(u))$ and $y = g_1(g_2(u))$. Of course, in the first case $u_2 = y_1$ and in the second one $u_1 = y_2$. Evidently, the systems (1) can be described by K_1 and K_2 which we think of as their transfer functions. This satisfies both the validity of the algebra of transfer functions and also the inclusion of nonlinearities into the transfer function.

3. PRELIMINARIES

The simple idea depicted in §2 can be similarly applied also to nonlinear control systems. The scope of our interest is restricted to situations in which the system properties are generic and to the systems defined by means of analytic or also meromorphic functions. Such an approach is widely discussed in (Conte *et al.*, 1999) and it considers an algebraic point of view in nonlinear systems. In this section some algebraic tools and methods are briefly reviewed. The reader is referred to (Conte *et al.*, 1999) for detailed technical constructions which are not found here.

The dynamic systems, mainly considered in this paper, are described by a system of first order differential equations of the form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= g(x,u) \end{aligned} \tag{4}$$

where entries of f and g are meromorphic functions, which we think of as elements of the quotient field of the ring of analytic functions, and $x \in R^n$, $u \in R^m$ and $y \in R^p$ denote state, input and output to the system. Let $\mathcal{K}$ denote the field of meromorphic functions of x, u and a finite number of derivatives of u. A derivative operator δ acting on $\mathcal{K}$ can be defined as follows

$$\begin{aligned} \delta F\left(x,u^{(k)}\right) &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\delta x_i + \sum_{\substack{j=1 \\ k\geq 0}}^{m} \frac{\partial F}{\partial u_j^{(k)}}\delta u_j^{(k)} \\ \delta x_i = \dot{x}_i &= f_i(x) + g_i(x)u \qquad i = 1,\ldots,n \\ \delta u_j^{(k)} &= u_j^{(k+1)} \qquad k \geq 0,\, j = 1,\ldots,m \end{aligned} \tag{5}$$

We often use a notation $\delta u = \dot{u}$ or $u^{(2)} = \ddot{u}$ etc. It is important to say that derivative operator δ satisfies $\delta^k(\mathrm{d}F) = \mathrm{d}\left(\delta^k F\right)$ for any $F \in \mathcal{K}$ and $k \geq 0$.

4. SYMBOLIC COMPUTATION FOR NONLINEAR SYSTEMS

The Laplace transformation plays an important role for linear systems. It enables to design control algorithms, to switch from an input-output description of a linear system to a state space representation, etc. (Conte *et al.*, 1999) states that such a symbolic computation is not available for nonlinear systems. The remarkable fact is that an algebraic point of view, reviewed in previous section, enables to define similar symbolic computation. This was firstly mentioned in (Halás *et al.*, 2003) where, instead of an algebraic approach, the velocity form of nonlinear systems was used.

4.1 State-space representation

Given the system (4), let us denote by $\mathcal{Y}$ the space defined by $\mathcal{Y} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y\}$. It follows from (4) that

$$\begin{aligned} \mathrm{d}\dot{x} &= A\mathrm{d}x + B\mathrm{d}u \\ \mathrm{d}y &= C\mathrm{d}x + D\mathrm{d}u \end{aligned} \tag{6}$$

where $A = (\partial f/\partial x)$, $B = (\partial f/\partial u)$, $C = (\partial g/\partial x)$ and $D = (\partial g/\partial u)$. Then $\mathcal{Y} = \mathrm{span}_{\mathcal{K}}\{C\mathrm{d}x + D\mathrm{d}u\}$. To find a symbolic computation, which would play a similar role to the Laplace transform for linear systems, dx has to be expressed in terms of du. Then the relationship between the output and the input space would be established. This cannot be done directly from (6). Nevertheless, equations (6) can be, by considering $\delta^k(\mathrm{d}F) = \mathrm{d}\left(\delta^k F\right)$, rewritten as follows

$$\begin{aligned} (\delta I - A)\mathrm{d}x &= B\mathrm{d}u \\ \mathrm{d}y &= C\mathrm{d}x + B\mathrm{d}u \end{aligned} \tag{7}$$

It follows from (7) that we can formally find $F(\delta)$ such that $\mathrm{d}y = F(\delta)\mathrm{d}u$. We think of $F(\delta)$ as an analogy of transfer functions of linear systems.

Example 1. Given the following system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1x_2 - \sin x_1 + u \\ y &= x_1 \end{aligned} \tag{8}$$

Differentiating gives

$$A = \begin{bmatrix} 0 & 1 \\ -x_2 - \cos x_1 & -x_1 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{9}$$

Compute the transfer function

$$F(\delta) = C(\delta I - A)^{-1}B = \frac{1}{\delta^2 + x_1\delta + \cos x_1 + x_2} \tag{10}$$

Note that "poles" depend on the state. In other words, coefficients of the transfer function are not just constants, they are meromorphic functions. That is the way how to include nonlinear dynamics into the transfer function.

4.2 Input-output description

Consider a SISO nonlinear system described by a differential equations of the form

$$F\left(y^{(n)},\ldots y,u^{(m)},\ldots,u\right)=0 \tag{11}$$

where F is a meromorphic function. We can assign to (11) a differential form

$$a_n \mathrm{d}y^{(n)}+\ldots+a_0\mathrm{d}y=b_m\mathrm{d}u^{(m)}+\ldots+b_0\mathrm{d}u \tag{12}$$

where $F\in\mathcal{K}$, $a_i=\partial F/\partial y^{(i)}$, $i=1,\ldots,n$ and $b_j=\partial F/\partial u^{(j)}$, $j=1,\ldots,m$. Equation (15) can be, by considering $\delta^k(\mathrm{d}F)=\mathrm{d}\left(\delta^k F\right)$, rewritten as

$$\left(a_n\delta^n+\ldots+a_0\right)\mathrm{d}y=\left(b_m\delta^m+\ldots+b_0\right)\mathrm{d}u \tag{13}$$

and, finally, the transfer function of (11) is given by

$$F(\delta)=\frac{b_m\delta^m+\ldots+b_0}{a_n\delta^n+\ldots+a_0} \tag{14}$$

Example 2. Consider the nonlinear system with dynamics

$$\ddot{y}+\sin y=u\dot{y}+u^2 \tag{15}$$

After differentiating

$$\begin{gathered}\mathrm{d}\ddot{y}+\cos y\mathrm{d}y=\dot{y}\mathrm{d}u+u\mathrm{d}\dot{y}+2u\mathrm{d}u\\ \left(\delta^3-u\delta+\cos y\right)\mathrm{d}y=\left(\dot{y}+2u\right)\mathrm{d}u\end{gathered} \tag{16}$$

the transfer function is given by

$$F(\delta)=C(\delta I-A)^{-1}B=\frac{\dot{y}+2u}{\delta^3-u\delta+\cos y} \tag{17}$$

5. APPLICATIONS

Here the defined symbolic computation is employed to depict a solution to a few basic control problems. Namely, a modeling and a control design.

5.1 Modeling

We are interested in switching between a state space representation and an input-output description of a nonlinear system. In that respect, the Laplace transformation is available for linear systems. The problem is analyzed in (Conte *et al.*, 1999), where is also reflected that such a symbolic computation is not available for nonlinear systems. However, the defined transfer functions can play a similar role.

Example 3. Consider again the nonlinear system (8) with the transfer function (10). The state x_1 and x_2 can be expressed as $x_1=y$ and $x_2=\dot{y}$. Then

$$\mathrm{d}y=F(\delta)\mathrm{d}u=\frac{1}{\delta^2+y\delta+\cos y+\dot{y}}\mathrm{d}u \tag{18}$$

Now, the input-output description is obtained as

$$\begin{gathered}\delta^2\mathrm{d}y+y\delta\mathrm{d}y+\cos y\mathrm{d}y+\dot{y}\mathrm{d}y=\mathrm{d}u\\ \mathrm{d}\ddot{y}+y\mathrm{d}\dot{y}+\cos y\mathrm{d}y+\dot{y}\mathrm{d}y=\mathrm{d}u\end{gathered} \tag{19}$$

The last expression represents a differential of the second order differential equation

$$\ddot{y}+y\dot{y}+\sin y=u \tag{20}$$

Notice the similarity to linear systems and Laplace transforms.

5.2 Control design

This section deals with the properties of the introduced transfer functions in the term of their algebra. Evidently, each complex system structure can be divided into three basic connections: series, parallel and feedback. To compute the final transfer function of that complex structure one has to know how to compute a transfer function of these basic connections.The reader is referred to (Halás *et al*., 2003) for constructions and proofs.

Example 4. The example presented here deals with the real application. We will consider a fluid tank plant described by the differential equation

$$\begin{aligned} \dot{x} &= \frac{1}{S}u - c\sqrt{x} \\ y &= x \end{aligned} \tag{21}$$

where x denotes a level of a fluid, S denotes a tank area and c denotes a flow coefficient. The real system was identified with: $S = 1{,}7.10^{-3}$, $c = 4{,}1.10^{-3}$. (parameters are specified without units since they include also properties of a pump and a sensor). We can assign to the system a transfer function as follows

$$F(\delta) = \frac{1/S}{\delta + \dfrac{c}{2\sqrt{x}}} \tag{22}$$

The aim is to design a controller, which satisfies a linear behaviour for both a control and also an input disturbance. Chosen feedback structure is depicted in Fig. 1.

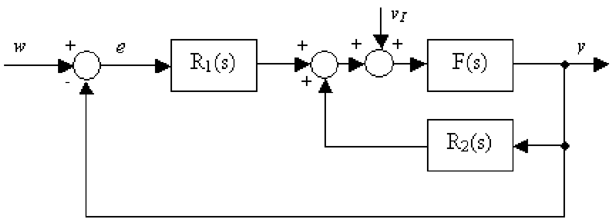

Fig. 1. Multiloop control structure.

Accordingly, the transfer functions of control and input perturbation can be computed as

$$G_{yw}(\delta) = \frac{R_1(\delta)F(\delta)}{1 + [R_1(\delta) - R_2(\delta)]F(\delta)} \tag{23}$$

$$G_{yv_I}(\delta) = \frac{F(\delta)}{1 + [R_1(\delta) - R_2(\delta)]F(\delta)} \tag{24}$$

The solutions to the set of equations (23) and (24) solved for $R_1(\delta)$ and $R_2(\delta)$ are

$$R_1(\delta) = \frac{G_{yw}(\delta)}{G_{yv_I}(\delta)} \tag{25}$$

$$R_2(\delta) = \frac{F(\delta)G_{yw}(\delta) - F(\delta) + G_{yv_I}(\delta)}{F(\delta)G_{yv_I}(\delta)} \tag{26}$$

Final transfer functions of controllers (19) and (20), with the control transfer function $G_{yw}(\delta)$ chosen to be $\dfrac{-\alpha}{\delta - \alpha}$ and the perturbation transfer function $G_{yv_I}(\delta)$ chosen to be $\dfrac{\delta}{(\delta - \alpha)^2}$, lead to the linear PI controller

$$R_1(\delta) = -\alpha + \frac{\alpha^2}{\delta} \tag{27}$$

and the nonlinear PD controller

$$R_2(\delta) = \delta(S-1) + \frac{Sc}{2\sqrt{x}} + \alpha \tag{28}$$

In the case the gain 1/S of the plant (22) is set to 1 the D action of controller (28) can be removed. This is able to do by inserting gain $K = S$ before system *F(δ)*. Then, it leads to the nonlinear state controller

$$R_2(\delta) = \frac{Sc}{2\sqrt{x}} + \alpha \tag{29}$$

The next step is to find differential equations concerning these transfer functions. One receives

$$\dot{u}_{R_1} = -\alpha\dot{e} + \alpha^2 e \;\; ; \;\; u_{R_2} = Sc\sqrt{x} + \alpha x \tag{30}$$

Final control law can be written in form

$$u = S(u_{R1} + u_{R2}) \tag{31}$$

Real system responses (solid line) with input perturbations in times 300s, 800s, 1300s and 1800s, and simulation results (doted line), both with the closed loop pole α chosen to be –0.02, are presented in Fig. 2. Responses correspond well to each other.

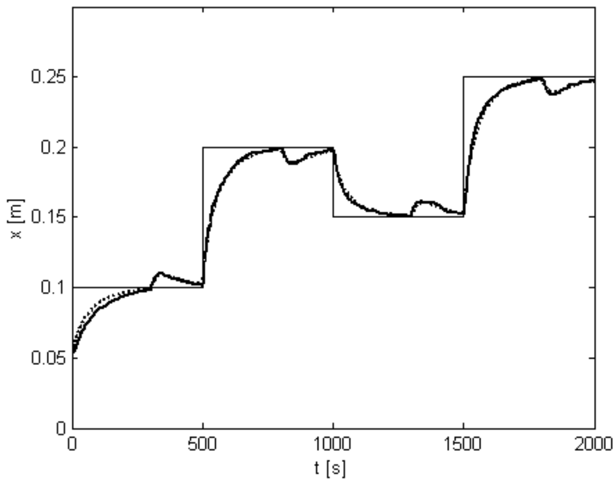

Fig. 2. Real system and model responses.

6. CONCLUSIONS

To conclude this paper it should be firstly noted that an algebraic point of view enables to define a symbolic computation for nonlinear systems with the possibility to design nonlinear controllers. The remarkable fact is that the algebra of the defined transfer functions is the same like in linear systems. However, it does not mean that all linear theory can be applied to nonlinear problems. It is important to compare presented symbolic computation to other techniques and methods of linear and nonlinear systems. There can be still many questions, for instance, on stability, controllability etc. which are, regrettably, out of scope of this paper.

REFERENCES

Conte, G., C.H. Moog and A.M. Perdon (1999). *Nonlinear Control Systems: An Algebraic Setting*, Vol. 242, Lecture Notes in Springer-Verlag, London.

Halás, M., M. Huba and K. Žáková (2003). The Exact Velocity Linearization Method. In: *Proc. of the 2nd IFAC Conference on Control System Design*. Bratislava, Slovakia.

Isidori A. (1989). *Nonlinear Control Systems*. Springer-Verlag, Berlin.